Analyse topologique des données

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Topologie: la sphère

Topologie: les tores

Figure: Tore avec 1 trou, 2 trous, 3 trous...trou=genre

Topologie

Définition (Vaseuse)

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Définition

On dit que 2 variétés topologiques sont les mêmes topologiquement si on peut passer d'une figure à l'autre avec une bijection continue et dont la réciproque est continue

$E = x^2 + y^2 + z^2 - 1$ et $F = 2.x^2 + 3.y^2 + 4.z^2 - 5$. Pour passer de E à F, on considère pour $t \in [0, 1]$

$$t.E + (1 - t).F$$

Homologie simpliciale

	sommets	arêtes	faces	S-A+F
tétrahèdre	4	6	4	4-6+4=2
cube	8	12	6	8-12+6=2
octaèdre	6	12	8	6-12+8=2

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Mais pourquoi ça fait 2 ?

Homologie simpliciale et nombre de Betti (1823-1892)

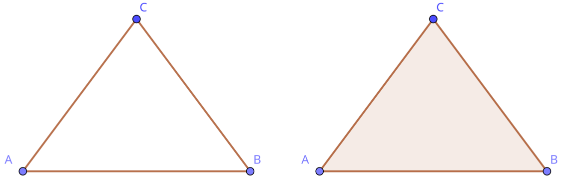
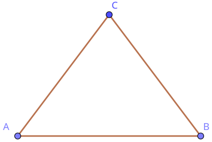


Figure: Triangle creux ou triangle plein

- dimension 0: On a 3 sommets $\rightsquigarrow \mathbb{R}^3 = \mathbb{R}.A \times \mathbb{R}.B \times \mathbb{R}.C$
- dimension 1: On a 3 arêtes $\rightsquigarrow \mathbb{R}^3 = \mathbb{R}.[AB] \times \mathbb{R}.[BC] \times \mathbb{R}[CA]$
- dimension 2: On a 1 face ou 0 selon qu'on regarde le triangle creux ou plein.

Nombre de Betti du triangle creux



$$\begin{array}{ccc} \text{dim } 1 & & \text{dim } 0 \\ \mathbb{R}^3 & \xrightarrow{\quad f \quad} & \mathbb{R}^3 \end{array}$$

$$[AB] = (1,0,0) \longrightarrow B - A = (-1,1,0)$$

$$[BC] = (0,1,0) \longrightarrow C - B = (0,-1,1)$$

$$[CA] = (0,0,1) \longrightarrow A - C = (1,0,-1)$$

$$= (C - B) - (B - A)$$

- On définit $H^0(\mathring{T}) = \mathbb{R}^3 / f(\mathbb{R}^3) = \mathbb{R}^1$ donc $b_0(\mathring{T}) = 1$
- On définit $H^1(\mathring{T}) = \{v \mid f(v) = 0\} = \mathbb{R}.(1,1,1)$ donc $b_1(\mathring{T}) = 1$.

Homologie simpliciale et nombre de Betti

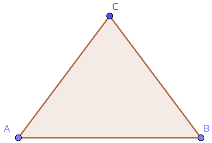


Figure: On considère le triangle T plein

$$\begin{array}{ccccc} \dim 2 & & \dim 1 & & \dim 0 \\ \mathbb{R} & \xrightarrow{\quad g \quad} & \mathbb{R}^3 & \xrightarrow{\quad f \quad} & \mathbb{R}^3 \end{array}$$

$$(ABC) \longrightarrow [BC] - [AC] + [AB] = (1, 1, 1)$$

- On définit $H^0(T) = \mathbb{R}^3 / f(\mathbb{R}^3) = \mathbb{R}$ donc $b_0(T) = 1$
- On définit $H^1(T) = \{v \mid f(v) = 0\} / g(\mathbb{R}) = 0$ donc $b_1(T) = 0$
- On définit $H^2(T) = \{v \mid g(v) = 0\} = 0$ donc $b_2(T) = 0$

Nombre de Betti et Caractéristiques d'Euler-Poincaré

	$\mathring{T}$	Cercle	T	disque	point
b_0	1	1	1	1	1
b_1	1	1	0	0	0
b_2	0	0	0	0	0
$\chi = b_0 - b_1 + b_2$	0	0	1	1	1

Définition

La somme alternée des nombres de Betti est appelée la caractéristique d'Euler-Poincaré et on la note χ.

Théorème

Si deux forment sont topologiquement identiques alors elles ont les mêmes nombres de Betti.

1. Un triangle creux est topologiquement comme un cercle.
2. Un triangle plein est topologiquement comme un point.

Nombre de Betti du tétraèdre

Théorème

On peut faire le même calcul sur un thétraèdre et on trouve

$$b_0(Tétraèdre) = 1$$
$$b_1(Tétraèdre) = 0$$
$$b_2(Tétraèdre) = 1$$

La caractéristique d'Euler-Poincaré est

$$\chi(Tétraèdre) = b_0 - b_1 + b_2 = 2$$

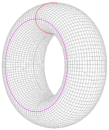
Comme le tétraèdre est topologiquement une sphère alors on comprend que le "2" qu'on obtenait est simplement la caractéristique d'Euler de la sphère.

Homologie simpliciale des tores

	Sphère	1-Tore	2-Tore	g-Tore
b_0	1	1	1	1
b_1	0	2	4	2g
b_2	1	1	1	1
$\chi = \sum(-1)^i b_i$	2	0	-2	2-2g

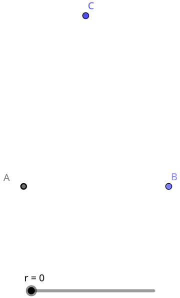
Théorème

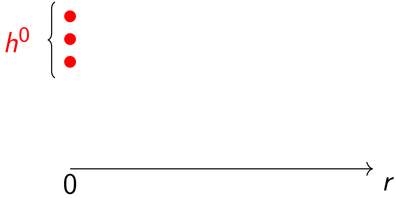
1. *On ne peut pas déformer une sphère en un tore.*
2. *Deux tores sont topologiquement identiques si et seulement si, ils ont le même genre càd le même nombre de trous.*



Comment utiliser la topologie pour analyser les données ?

Homologie persistante





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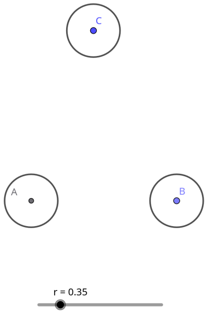

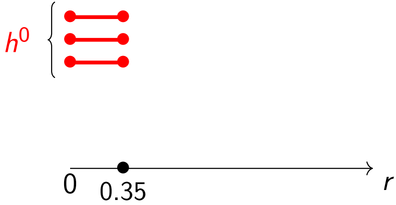

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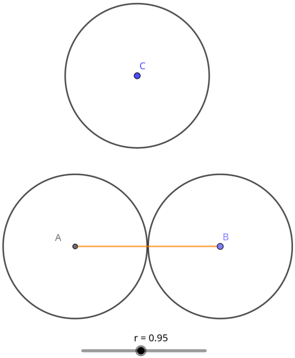

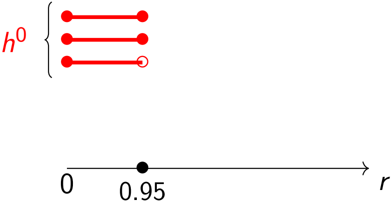

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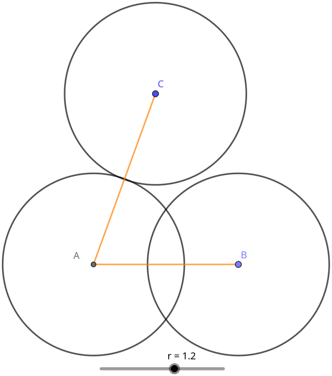

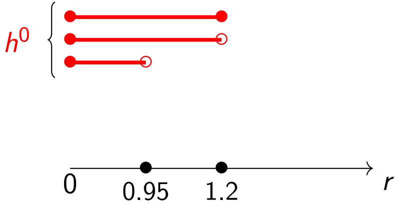

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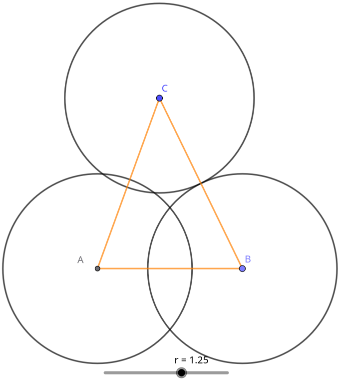

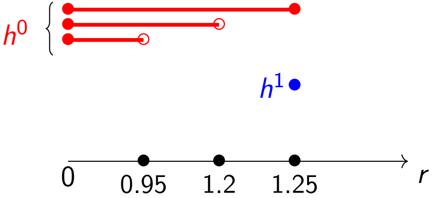

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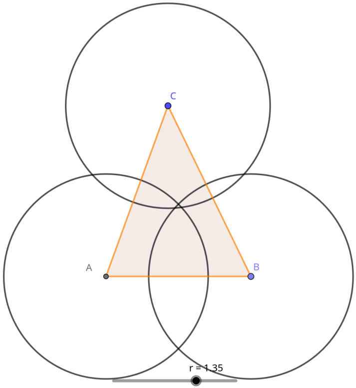

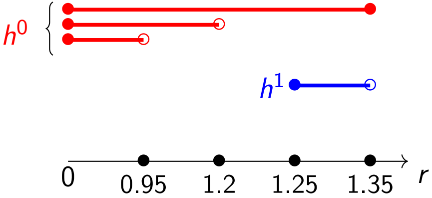

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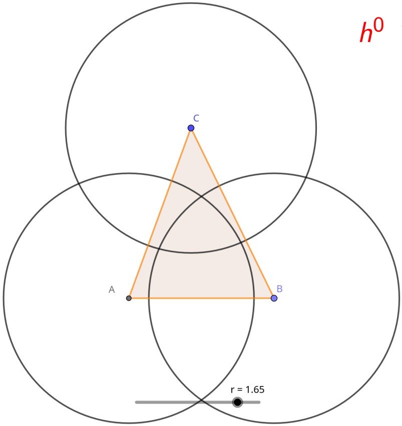

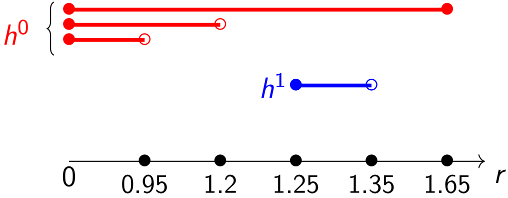

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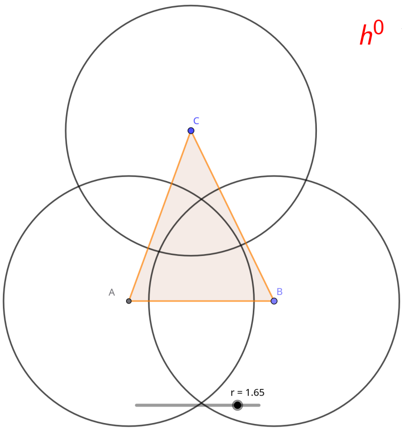


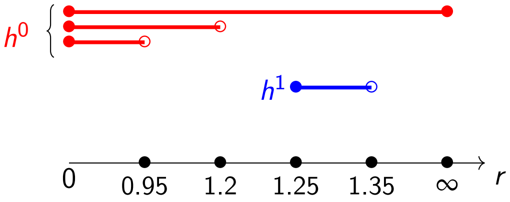
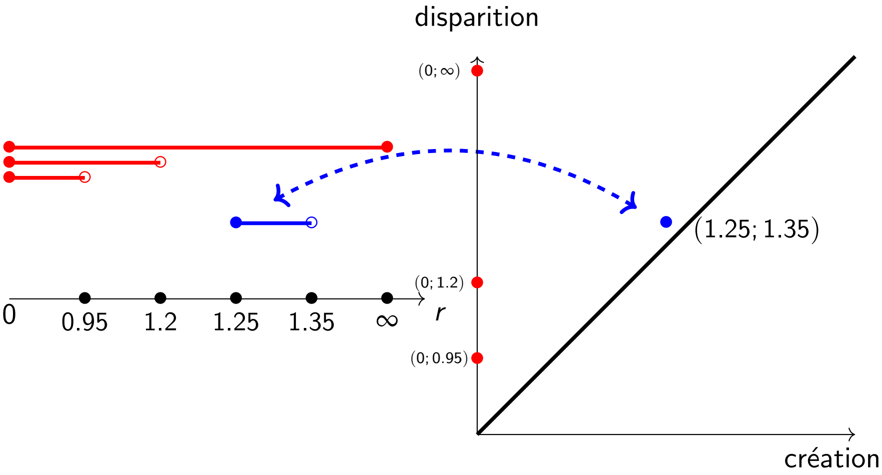
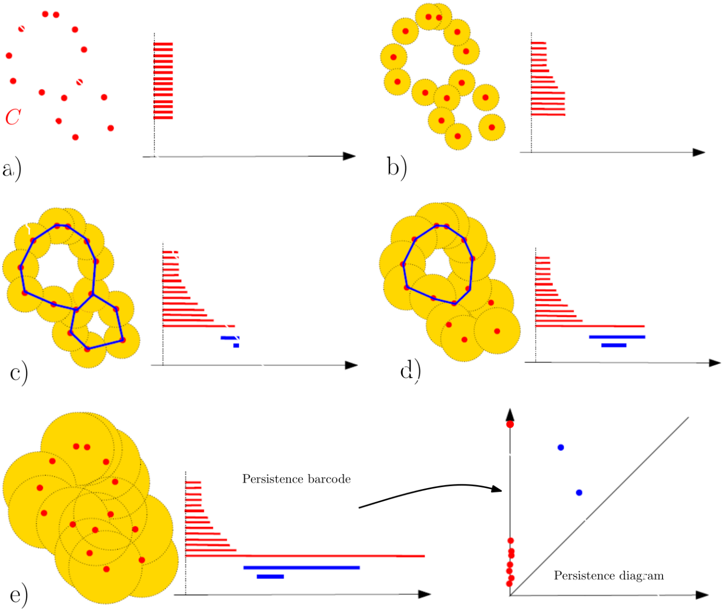


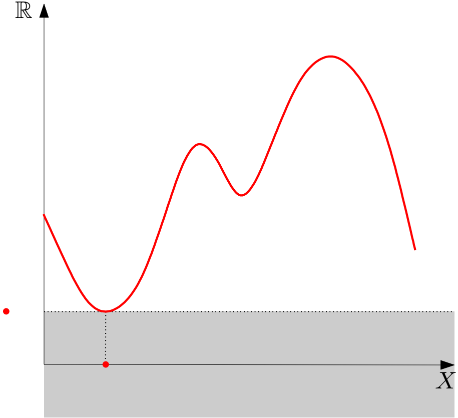
Diagramme de persistante




C
a)
b)
c)
d)
e)
Persistence barcode
Persistence diagram

Persistent homology for functions

- Tracking and encoding the evolution of the connected components (0-dimensional homology) of the sublevel sets of a function
- The family of sublevel sets of a function is an example of filtration.



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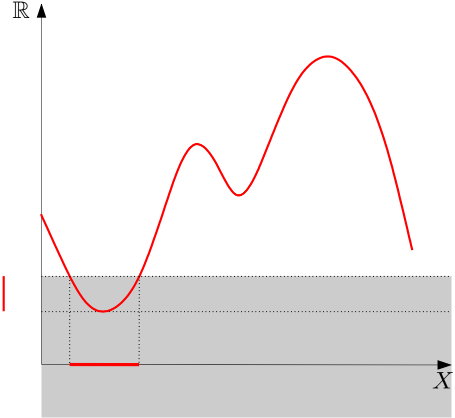

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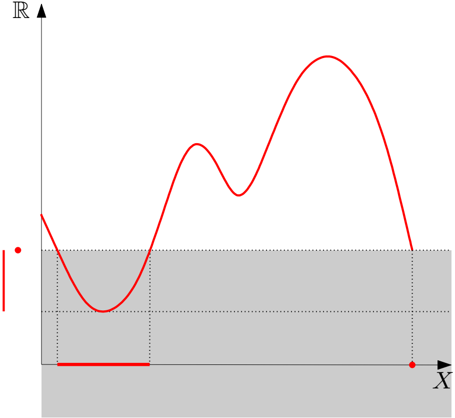

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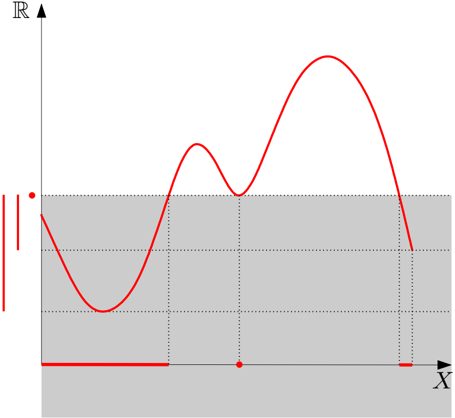

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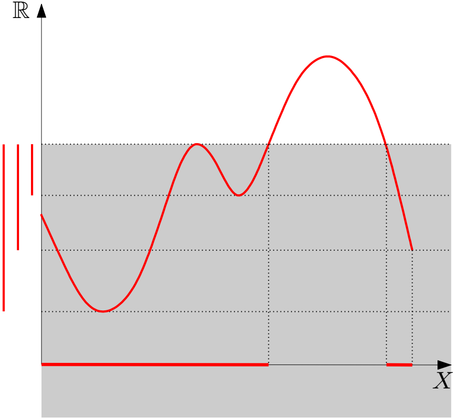

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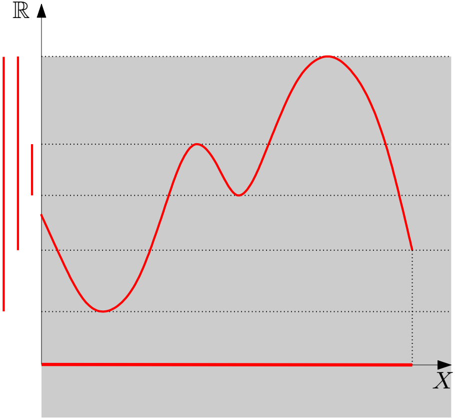

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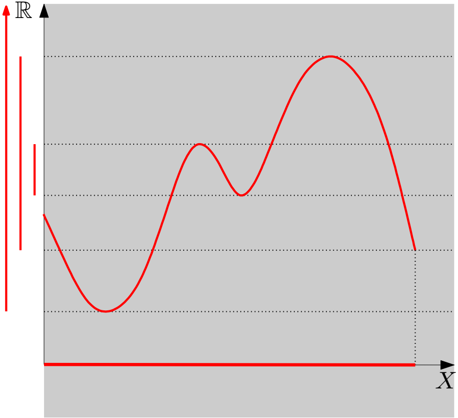

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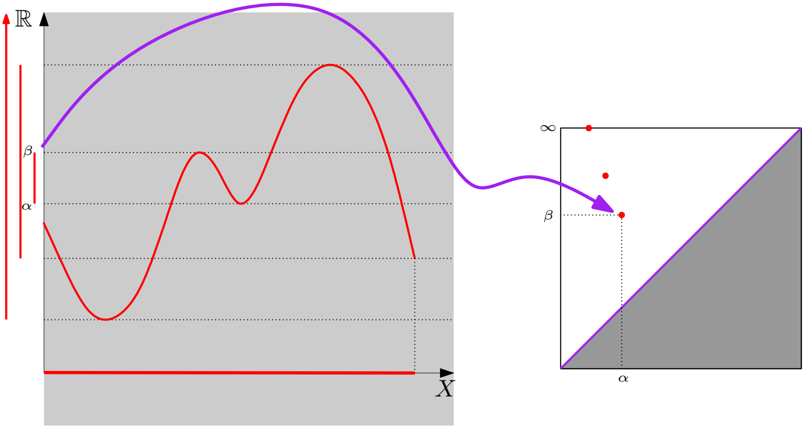
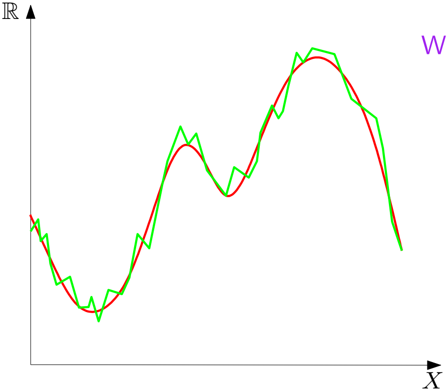
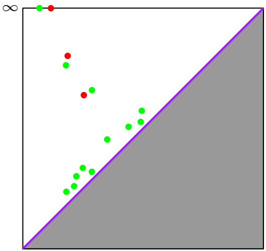


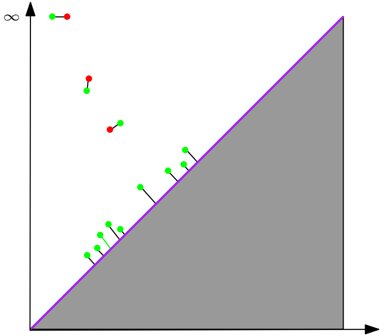
Stability properties



What if f is slightly perturbed?



Distance between persistence diagrams



The bottleneck distance between two diagrams D_1 and D_2 is

$$d_B(D_1, D_2) = \inf_{\gamma \in \Gamma} \sup_{p \in D_1} \|p - \gamma(p)\|_\infty$$

where Γ is the set of all the bijections between D_1 and D_2 and $\|p - q\|_\infty = \max(|x_p - x_q|, |y_p - y_q|)$.

Important Remark: There is one persistence diagram per homology dimension. In general, are compared diagrams corresponding to same homology dim.